Problems at the Quantum/Classical Interface

Nicholas Wheeler, Reed College Physics Department
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Introduction. Much of my work for the past year or more has radiated from my interest in discovering the answers to a nest of interrelated questions that arise at the place where quantum mechanics and classical mechanics join. But my work has proceeded very nonlinearly, down paths that have branched, and branched again . . . with the result that it would not be obvious to a casual reader—is sometimes difficult even for me to recall—what the detailed work of the moment has to do with larger issues. My intention here is to state what I imagine those issues to be, and to describe the difficulties I have encountered (and at the moment have me stopped cold in my tracks).

Why does the macroscopic world look classical? The question might be summarily dismissed on grounds that this coin, this leaf . . . "look classical" only to those unable to see (with the mind's eye) the solid state physics, the quantum chemistry implicit in their gross attributes. I grant the point, accept that *every property of every thing* that intrudes upon my consciousness is quantum mechanical (and that so also must be the operation of mind itself). My problem lies elsewhere.

Classical objects are, if at one place, not at another. The question is: *How does the wavefunction $\psi(x,t)$, if it refers to such an object, come to be—not by the contrivance of an observer, but spontaneously—spatially localized*?

In late 1953 and early 1954, Einstein and Max Born debated the question.[1] Einstein had objected that the quantum mechanical account of what he called the "ball-between-walls" problem did not reproduce the classical physics of that simple system, so could not be complete. Born's response was that one had only to *select the right kind of quantum state* and all would be as expected. To which Einstein countered

> . . . *I see, however, that you want to relate classical mechanics only to those ψ-functions which are narrow with respect to coordinates and momenta. But when one looks at it in this way, one could come*

[1] *The Born-Einstein Letters* (1971), Letters 105–116. The exchange involved Pauli (who sided with Born) in its later stages.

> *to the conclusion that macro-mechanics cannot claim to describe, even approximately, most of the events in macro-systems that are conceivable on the quantum theory. For example, one would then be very surprised if a star, or a fly seen for the first time, appeared even to be quasi-localized.*

Born considers Einstein's position to be evidence of his "inadequate knowledge of quantum mechanics," and fails, it seems to me, to meet the force of Einstein's observation. Born (in collaboration with W. Ludwig: Zeitschrift für Physik **150**, 106 (1958)) was inspired to construct a detailed technical demonstration of his position (which he is certain "every quantum theoretician would . . . recognize . . . [to be] correct"), but I have been surprised to discover that his mathematics does, in fact, not support his conclusions. The 'particle-in-a-box" problem is treated near the beginning of most introductory quantum texts, and provides a natural laboratory for exploring the some aspects of the questions that concern me, but close analysis reveals details that I find highly counterintuitive.

Wavepacket dispersal for an unrestrained free particle. Solutions

$$\psi(x,t;p) \equiv \exp\left\{\tfrac{i}{\hbar}\left(px - \tfrac{1}{2m}p^2 t\right)\right\} \equiv e^{i(kx-\omega t)}$$

of the time-dependent Schödinger equation

$$-\tfrac{\hbar^2}{2m}\psi_{xx} = i\hbar\psi_t$$

must are not normalizable, so must be taken in normalized linear combinations

$$\psi(x,t) = \tfrac{1}{\sqrt{h}}\int \varphi(p)\exp\left\{\tfrac{i}{\hbar}\left(px - \tfrac{1}{2m}p^2 t\right)\right\}dp$$

called wavepackets: by Parseval's theorem one then has

$$\int \psi^*(p)\psi(p)\,dp = \int \varphi^*(p)\varphi(p)\,dp$$

and requires of $\varphi(p)$ that $\int \varphi^*\varphi\,dp = 1$.

In textbooks it is standard to look the (surprisingly intricate) example that springs from the assumption that initially

$$|\psi(x,0)|^2 = \tfrac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left[\frac{x-a}{\sigma}\right]^2} \quad : \quad \text{Gaussian}$$

but more general conclusions can be obtained by simpler means. Look to

$$\begin{aligned}
\tfrac{d}{dt}\langle \mathsf{x}\rangle &= \tfrac{1}{i\hbar}(\psi|[\mathsf{x},\mathsf{H}]|\psi) = \tfrac{1}{m}\langle \mathsf{p}\rangle\\
\tfrac{d}{dt}\langle \mathsf{p}\rangle &= \tfrac{1}{i\hbar}(\psi|[\mathsf{p},\mathsf{H}]|\psi) = 0
\end{aligned}$$

which—for *all* wavepackets—supplies

$$\langle \mathsf{p} \rangle_t = \text{constant; call it } p_0$$
$$\langle \mathsf{x} \rangle_t = x_0 + \tfrac{1}{m} p_0 t$$

So in all cases the mean drifts uniformly. From the latter of the preceding equations it follows, by the way, that

$$\tfrac{d^2}{dt^2}\langle \mathsf{x} \rangle^2 = \tfrac{2}{m^2}\langle \mathsf{p} \rangle^2$$

which will find use in a moment.

Look next to the motion of the second moments: we have

$$\tfrac{d}{dt}\langle \mathsf{x}^2 \rangle = \tfrac{1}{m}\langle \mathsf{x}\,\mathsf{p} + \mathsf{p}\,\mathsf{x} \rangle$$
$$\tfrac{d}{dt}\langle \mathsf{x}\,\mathsf{p} + \mathsf{p}\,\mathsf{x} \rangle = \tfrac{2}{m}\langle \mathsf{p}^2 \rangle$$
$$\tfrac{d}{dt}\langle \mathsf{p}^2 \rangle = 0$$

giving

$$\tfrac{d^2}{dt^2}\langle \mathsf{x}^2 \rangle = \tfrac{2}{m^2}\langle \mathsf{p}^2 \rangle \quad : \quad \text{non-negative constant}$$

We now have

$$\sigma_p^2 \equiv \langle \mathsf{p}^2 \rangle - \langle \mathsf{p} \rangle^2 = \langle (\mathsf{p} - \langle \mathsf{p} \rangle)^2 \rangle \quad : \quad \text{non-negative constant}$$

and the information that in all cases

$$\sigma_x^2 \equiv \langle \mathsf{x}^2 \rangle - \langle \mathsf{x} \rangle^2 = \langle (\mathsf{x} - \langle \mathsf{x} \rangle)^2 \rangle \quad : \quad \text{non-negative function of } t$$

satisfies

$$\tfrac{d^2}{dt^2}\sigma_x^2 = \tfrac{2}{m^2}\sigma_p^2$$

Therefore $\sigma_x^2(t) = \frac{1}{m^2}\sigma_p^2 t^2 + at + b$ which can be written

$$\sigma_x^2(t) = \sigma^2 + \left[\tfrac{1}{m}\sigma_p(t - t_0)\right]^2 \tag{1}$$

where σ is the *least* value ever assumed by $\sigma_x(t)$. This result contains no $\hbar$'s, and is in fact entirely classical—a consequence of the distinctive phase flow of a classical free particle. It becomes quantum mechanical when we appeal to the uncertainty principle to write

$$\sigma \cdot \sigma_p \geqslant \tfrac{1}{2}\hbar$$

Set $\sigma_p = \hbar/2\sigma$ to obtain

$$\sigma_x^2(t) = \sigma^2\Big\{1 + \big[(t - t_0)/\tau\big]^2\Big\} \quad \text{with} \quad \tau \equiv 2m\sigma^2/\hbar$$

From the hyperbolic design of this equation we see that the growth of $\sigma_x(t)$ is initially quadratic, but becomes asymptotically linear, proceeding with speed

$$v = \sigma/\tau = \hbar/2m\sigma$$

Observe that the characteristic dispersion time $\tau \uparrow \infty$, and that $v \downarrow 0$, in the "classical limit" $\hbar \downarrow 0$. If $\sigma = \hbar/mc$ (the "Compton length" of m) then $v = \frac{1}{2}c$. If $m\sigma^2 \sim 10^{-10}$ gm·cm$^2 = 10^{-10}$erg·sec^2 (which from a macroscopic standpoint is fairly "small") then $\tau \sim 10^{17}$sec, which is roughly the age of the universe. Note, however, that generally

$$\tau = m\sigma/\sigma_p$$

and if one retreats from $\sigma_p = \hbar/2\sigma$ to $\sigma_p > \hbar/2\sigma$ then (counterintuitively?) τ is *decreased*.

It was on such grounds—by appeal to the "cosmic [time] scale"—that Oppenheimer (somewhat condescendingly, it seems to me) sought to dismiss Einstein's criticism. But Einstein was unmoved: in a letter to Born dated 3 December 1953 Einstein writes

> *But one could easily quote some quite pedestrian examples where the divergence time is not all that long. I consider it too cheap a way of calming down one's scientific conscience. All the same it is not difficult to regard the step into probabilistic quantum theory as final. One only has to assume that the ψ-function relates to an ensemble, and not to an individual case; then one can use my example to describe, with the expected approximation (statistically conclusive), what classical mechanics also describes. According to the interpretation which you support in your letter, one has to regard this circumstance as a kind of coincidence. The interpretation of the ψ-function as relating to an ensemble also eliminates the paradox that a measurement carried out in* one *part of space determines the* kind *of expectation for a measurement carried out later in* another *part of space (couling of parts of systems far apart in space).*
>
> *One can safely accept the fact that, according to this concept, the description of a single system is incomplete, if one assumes that there is no correspondingly complete law for the complete description of the single system which determines its development in time ...*

Why localization? The argument developed on the preceding page (which extends naturally to higher moments) is almost *too* powerful, for it pertains to *all* wavepackets—even to wavepackets which present a pimple here and another a thousand miles away. It supplies the information that

$$\sigma_x(t) \text{ grows slowly when } m\sigma^2 \text{ is large}$$

but provides no insight into why, when m is large, we can expect ψ to present a pimple here and *none* far away; no insight into why

$$\text{large } m \text{ implies spontaneous localization(?)}$$

One might be tempted to argue that "large masses" are the fossilized product of prior "aggregation reactions"

$$A + B + C \longrightarrow AB + C \longrightarrow ABC$$

and that "aggregation presumes spatial proximity." But quantum mechanically one expects to have something like

$$|A+B+C) \longrightarrow |A+B+C)+|AB+C) \longrightarrow |A+B+C)+|AB+C)+|ABC)$$

and standard theory provides no means to achieve

$$|A+B+C)+|AB+C)+|ABC) \xrightarrow[\text{projective}]{} |ABC)$$

"spontaneously" (*i.e.*, without the participation of an "observer").

The localization problem appears on the grounds just sketched to be bound up with the problem of measurement, with the interpretation of ψ.

David Griffiths remarks that one expects to have

$$|A+B) \longrightarrow |AB) + \text{radiation}$$

One expects, in other words, to purchase localization at cost of de-localization (entropy production).

The problem is that it is by no means obvious how one is to go about the construction of *tractable detailed illustrations* of the physical principles to which I have alluded. It's hard to turn vague words (already too abundant in this subject area) into sharp calculations (too rare).

Confined free wavepackets. Installation of box boundary conditions at $x = 0$ and $x = a$ entails

$$\begin{aligned}\psi(x,0) &= \tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(k)e^{ikx}\,dk\\ &\downarrow\\ &= \sqrt{\tfrac{2}{a}}\sum_{n=1}^{\infty} f_n \sin k_n x \quad \text{with} \quad 0 \leqslant x \leqslant a \quad \text{and} \quad k_n \equiv n\tfrac{\pi}{a}\end{aligned}$$

and is often considered to simplify the analytical problem,[2] though my recent experience suggests otherwise. Two alternative viewpoints are available

- $\psi(x)$ continues periodically beyond the bounds of the box
- $\psi(x)$ vanishes beyond the bounds of the box

but lead (so it is standardly imagined) to identical physics. "Turning on time" produces

$$\psi(x,t) = \sum_{n=1}^{\infty} f_n e^{-i\omega_n t}\sin k_n x \quad \text{with} \quad \omega_n \equiv \tfrac{\hbar}{2m}k_n^2 = E_n/\hbar$$

Einstein inspired Born and his assistant (Ludwig) to undertake a study of the motion of *confined* wavepackets—though Einstein himself thought it

> *. . . obvious, even without a 'mathematical microscope', that the position must become more and more diffuse in the course of time. The one-dimensional case is similar, as the group-velocity depends on the wavelength. I think it would be a pity to waste your assistant's time when the result can never be in doubt. But if you are not convinced, by all means have the calculations done.*

I think it fair to say that such calculation struck Einstein as beside the point, since it proceeded from a carefully *selected* $\psi(x,0)$, and thus did not touch on what he considered to be the fundamental issue: the fabrication of a quantum world-as-we-find-it, without the participation of an active "observer."

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[2] See L. Schiff, *Quantum Mechanics* (3rd edition 1968), page 47.

Born reported to Einstein on 23 December 1953 that he found the detailed calculation "not at all easy, and I really had to rack my brain." The next April Pauli, who was then a visitor at the Institute for Advanced Study and had been drawn into the debate, remarked to Born that he had "used the example of a mass point between two walls, and of the wavepackets which belong to it, in my lectures in such a way that a transformation formula of the theta-functions comes into play ... but that is a mere detail." Born states that is was Pauli's casual reference to theta functions that inspired the effort that culminated in the Born/Ludwig paper (1958).

It is my present belief that the Pauli/Born/Ludwig analysis, though technically quite interesting, *fails to support the conclusions they so plausibly draw from it*. But before I describe the problems which lead me to that view I review the elementary classical physics that lends "plausibility" to their conclusions—physics upon which Einstein/Pauli/Born *et al* are in agreement.

Confined free classical ensembles. Pictures tell the story most simply. In Figure 1 we use the "method of images" to construct an account of the dispersal of an ensemble that departs the neighborhood Δx of x with velocities in the neighborhood Δv of v. Or alternatively: of what me might at time t estimate to be the position of a *single* particle which departs from an imperfectly known position with imperfectly defined velocity. Such a particle (alternatively: each of the members of such an ensemble) experiences periodic speed-preserving reversals of the sign of its velocity.

The figure suggests that the initial uncertainties are described by *flat*[3] distribution functions, but no such assumption is essential to the physics. Were one to write

$$\begin{aligned} p(x) &\equiv \text{initial } x\text{-distribution:} \quad 0 \leqslant x \leqslant a \\ q(v) &\equiv \text{initial } v\text{-distribution:} \; -\infty \leqslant v \leqslant +\infty \end{aligned}$$

then it would become natural to ask: What, in consequence of the equations of motion, can be said of $p(x,t)$? of $q(v,t)$? Those are by nature *marginal* distributions

$$\begin{aligned} p(x,t) &= \int F(x,v;t)\,dv \\ q(v,t) &= \int F(x,v;t)\,dx \end{aligned}$$

so the deeper question is: What, given $p(x,0)$ and $q(v,0)$, can be said of the dynamically evolved joint distribution $F(x,v;t)$? of how confinement affects the relation of $F(x,v;t)$ to $F(x,v;0)$?

Useful new light is cast on the situation when one looks (Figure 2) to the *phase flow* of such an ensemble. And this—as Figures 3 & 4 demonstrate—is most easily accomplished if one adopts a phase space analog of the method of

[3] By "flat" I here mean "constant *where non-zero*."

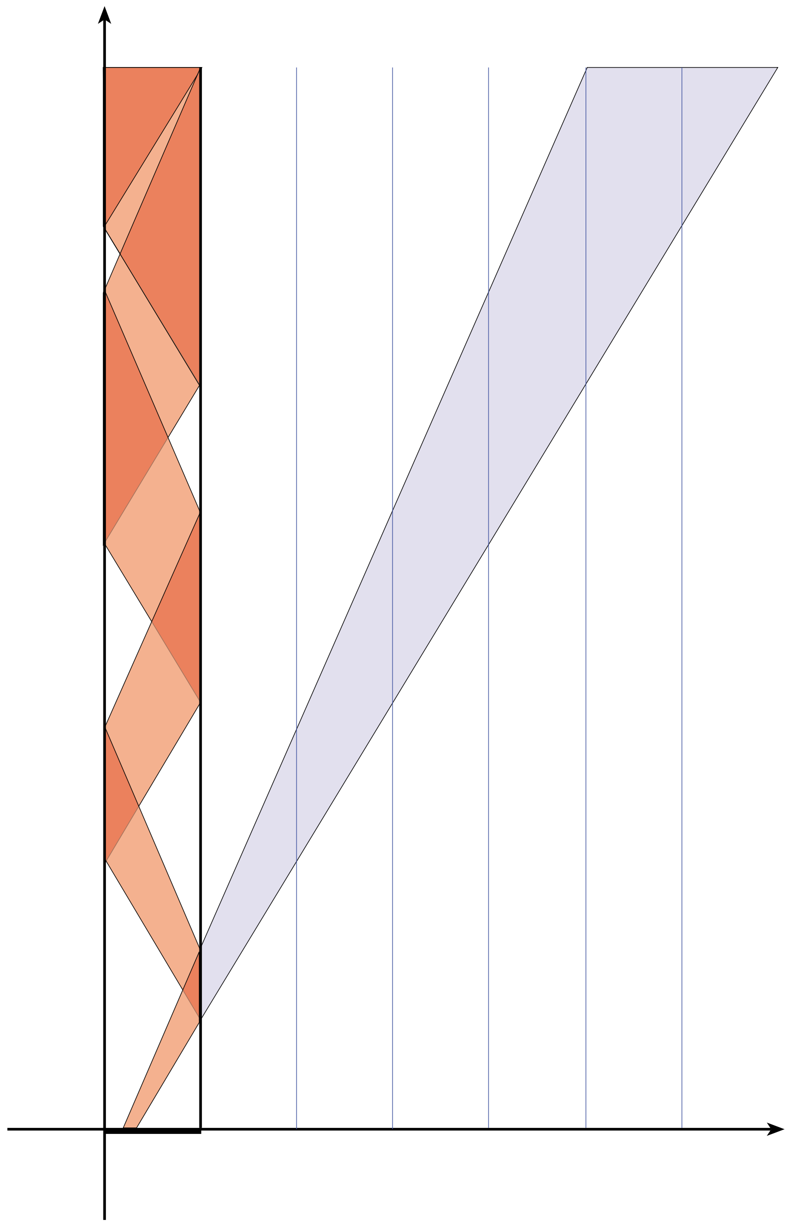

Figure 1: *Spacetime diagram of an ensemble that departs from the neighborhood* Δx *of* x *with velocities in the neighborhood* Δv *of* v*. In a time of the order*

$$\tau \sim \frac{\text{box width}}{\Delta v}$$

the ensemble has diffused enough to span the box.

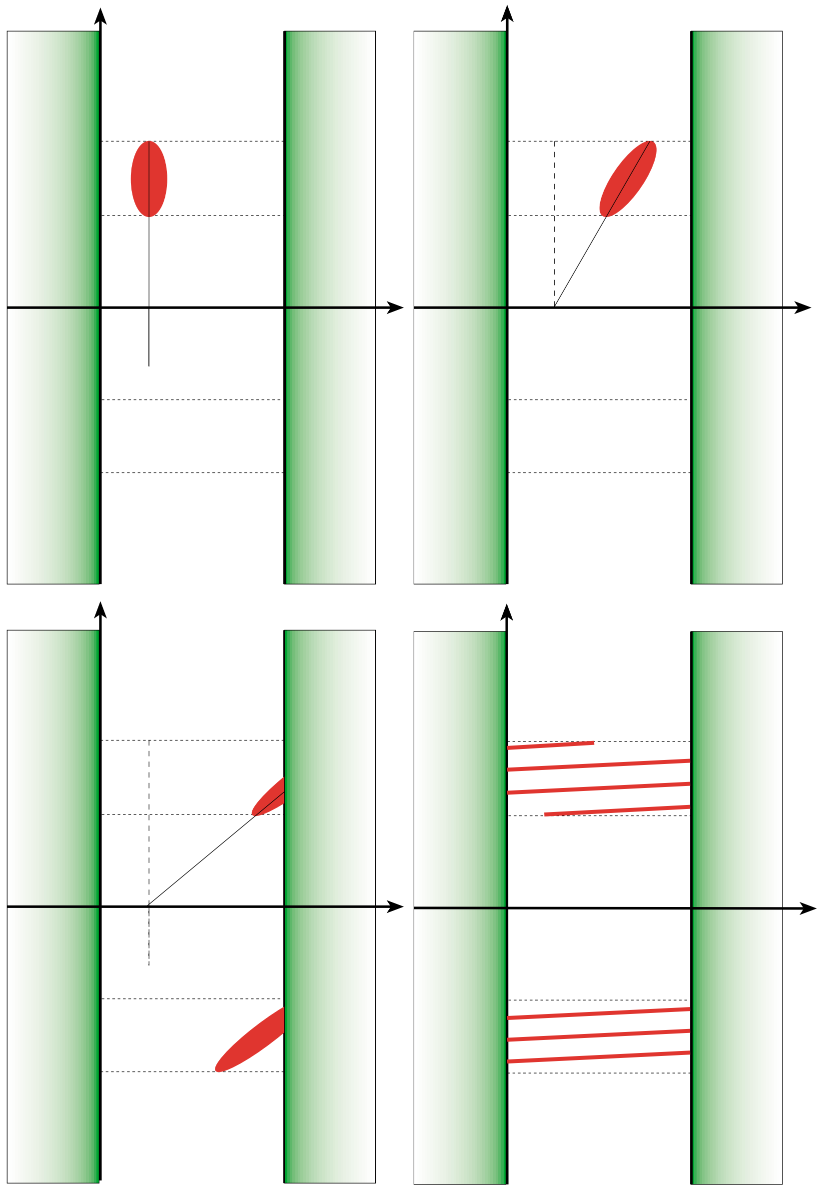

FIGURE 2: *We are now in phase space: the x-axis runs $\rightarrow$, the p-axis runs $\uparrow$. The initial distribution (upper left) has after a short time become distorted (upper right). A bit later (lower left) the faster elements have already hit the right wall and reversed the direction of their motion (sign of p), while the slower elements have yet to make first impact. Ultimately the distribution becomes striated; the striations become thinner and more closely spaced as their slope decreases.*

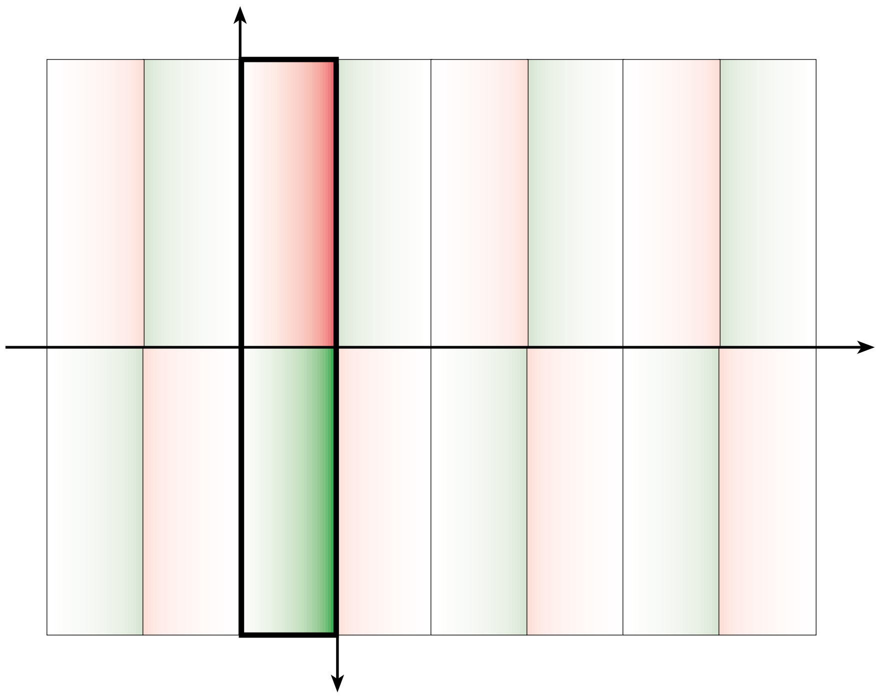

FIGURE 3: *The physics shown in Figure 3 is easier to conceptualize when the physical phase space is reflectively replicated. Here the red areas have $p > 0$, the green areas have $p < 0$ and the color gradient indicates the direction of increasing x. A heavy black line bounds the primary cell.*

images. Figure 2 and (more clearly) Figure 4 show how it comes about that—unless the initial distribution was monoenergetic—for $t \gg \tau$ the momentum distribution becomes "striated," and that the striations become progressively finer as t increases.

One exects (in the company, I think, of Einstein/Pauli/Born *et al*) to have

$$\begin{aligned}
\text{initial energy distribution} &\xrightarrow[t\to\infty]{} \text{unchanged}\\
\text{initial momentum distribution } u(p) &\xrightarrow[t\to\infty]{} \tfrac{1}{2}\big\{u(p)+u(-p)\big\} \;:\; \text{even}\\
\text{initial position distribution } p(x) &\xrightarrow[t\to\infty]{} \tfrac{1}{\text{box width}} \;:\; \text{flat}
\end{aligned}$$

but the situation cannot be quite that simple. For *time-reversal must in all cases serve to restore the original distribution.* So all the distinguishing detail that was written into the initial distribution must remain somehow intact, even asymptotically, and must serve to distinguish "flat from flat." I am motivated by this remark to inquire more carefully into the details of the mechanism by which $P(x, p; t) \equiv F(x, p/m; t)$ is assembled from the data written into

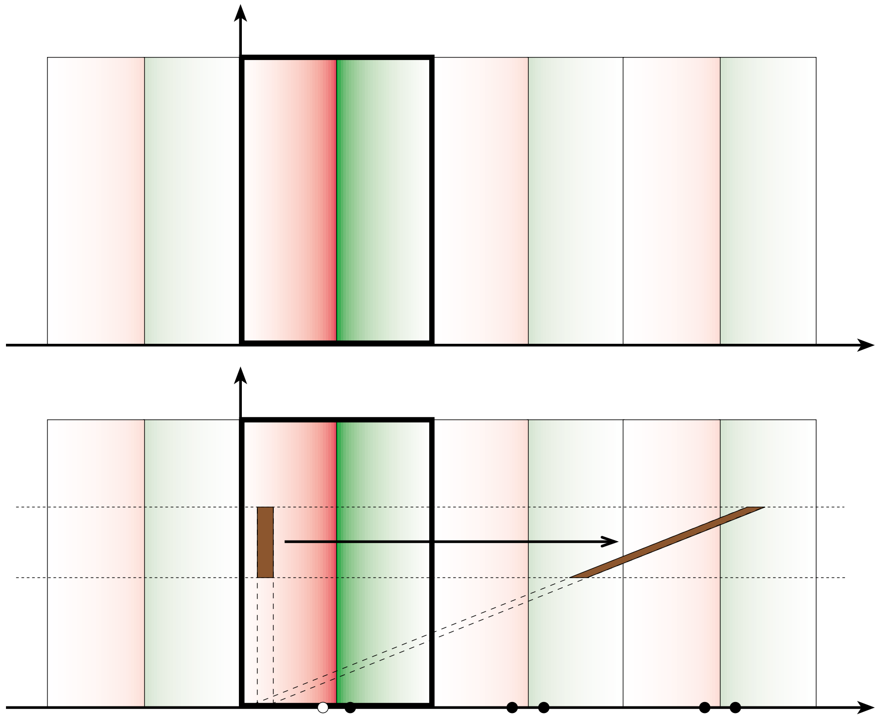

FIGURE 4: *Above, a more convenient variant of the preceding figure. The lower half of that figure has been abandoned, and the primary cell enlarged in compensation. Below is a representation of the dynamical flow experienced by an initially rectangular distribution. The black dots • all refer to the same spacepoint x. As t increases the slope of the evolved parallelogram decreases, and the number of momenta represented at x increases without limit . . . which was the main lesson of Figure 2.*

$P(x, p; 0) \equiv P(x, p)$. This is largely a matter of careful bookkeeping, and the variable of most immediate interest is again not p but its kinematic counterpart $v \equiv p/m$. Particles arrive at the spacetime point $\{x, t\}$ (x interior to the box) with a variety of velocities (momenta), and from v one can infer

- the point y from which the particle was launched at time $t = 0$
- the specific sequence—something like **RLRLR**—of its wall encounters.

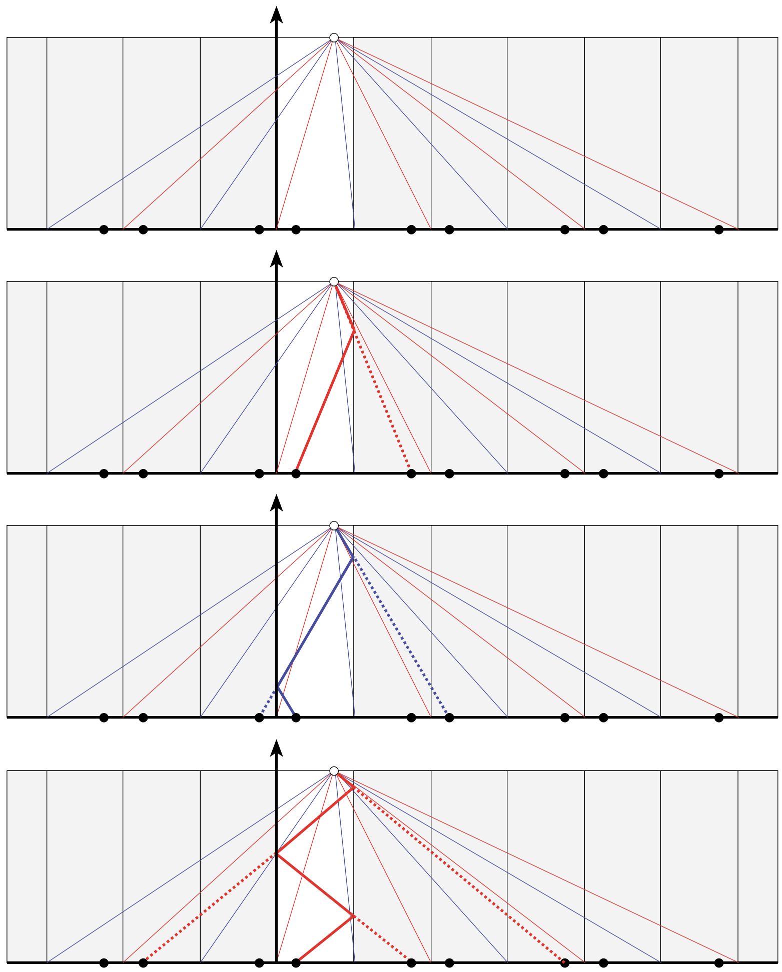

FIGURE 5: *Hall of mirrors:* ∘ *marks the spacetime point* $\{x,t\}$ *and* • *marks (reflective images of) the launch point* $\{y,0\}$. *The red/blue lines separate the*

- *direct path* $\{y,0\} \to \{x,t\}$ *from*
- *paths of type* **R** *(one shown in red) from*
- *paths of type* **LR** *(one shown in blue) from*
- *paths of type* **RLR** *(one shown in red) from . . .*

Working from Figure 5, we find that

$$\begin{aligned}
\text{direct paths arrive with velocities } v &= (x-y)/t\\
\text{paths of type } \textsf{\textbf{R}} \text{ arrive with velocites } v &= [x-(2a-y)]/t\\
\text{paths of type } \textsf{\textbf{LR}} \text{ arrive with velocites } v &= [x-(2a+y)]/t\\
\text{paths of type } \textsf{\textbf{RLR}} \text{ arrive with velocites } v &= [x-(4a-y)]/t\\
\text{paths of type } \textsf{\textbf{LRLR}} \text{ arrive with velocites } v &= [x-(4a+y)]/t\\
&\vdots
\end{aligned}$$

while

$$\begin{aligned}
\text{paths of type } \textsf{\textbf{L}} \text{ arrive with velocites } v &= (x+y)/t\\
\text{paths of type } \textsf{\textbf{RL}} \text{ arrive with velocites } v &= [x+(2a-y)]/t\\
\text{paths of type } \textsf{\textbf{LRL}} \text{ arrive with velocites } v &= [x+(2a+y)]/t\\
\text{paths of type } \textsf{\textbf{RLRL}} \text{ arrive with velocites } v &= [x+(4a-y)]/t\\
\text{paths of type } \textsf{\textbf{LRLRL}} \text{ arrive with velocites } v &= [x+(4a+y)]/t\\
&\vdots
\end{aligned}$$

where it is understood that in all cases $0 < y < a$. It now follows that

$$\begin{aligned}
p(x,t) = \quad & \int_0^a P(y, m\tfrac{x-y}{t}, 0)\tfrac{m}{t}dy + \sum_{n=1}^{\infty}\int_0^a P(y, m\tfrac{x-(2na-y)}{t}, 0)\tfrac{m}{t}dy\\
& \qquad + \sum_{n=1}^{\infty}\int_0^a P(y, m\tfrac{x-(2na+y)}{t}, 0)\tfrac{m}{t}dy\\
& + \int_0^a P(y, m\tfrac{x+y}{t}, 0)\tfrac{m}{t}dy + \sum_{n=1}^{\infty}\int_0^a P(y, m\tfrac{x+(2na-y)}{t}, 0)\tfrac{m}{t}dy\\
& \qquad + \sum_{n=1}^{\infty}\int_0^a P(y, m\tfrac{x+(2na+y)}{t}, 0)\tfrac{m}{t}dy\\
= & \sum_{n=-\infty}^{\infty}\int_0^a P(y, m\tfrac{x+y+2na}{t}, 0)\tfrac{m}{t}dy \qquad\qquad (2)\\
& \qquad + \sum_{n=-\infty}^{\infty}\int_0^a P(y, m\tfrac{x-y+2na}{t}, 0)\tfrac{m}{t}dy
\end{aligned}$$

Notice the spatial periodicity

$$p(x,t) = p(x+2a,t)$$

that springs spontaneously into being even if we assume $P(x,p;0)$ to vanish outside the box and to be aperiodic on its interior.

Figure 5 refers to constructions that make no sense in the limit $t \downarrow 0$, so we are not surprised to observe that neither does the formula that we have based upon those constructions. I postpone discussion of the opposite limit $t \uparrow \infty$ (in which we do have a lively interest) in order to examine some of the more immediate implications of (2).

We expect to have

$$\int_0^a p(x,t)\,dx = 1 \quad : \quad \text{all } t$$

while the right side of (2) supplies (for $t > 0$)

$$\begin{aligned}\int_0^a p(x,t)\,dx = \int_0^a \Big\{ \Big[\cdots + \int_{-2\alpha}^{-\alpha} + \int_0^{\alpha} + \int_{2\alpha}^{3\alpha} + \cdots \Big] P(y, p+q; 0)\,dp \\ + \Big[\cdots + \int_{-2\alpha}^{-\alpha} + \int_0^{\alpha} + \int_{2\alpha}^{3\alpha} + \cdots \Big] P(y, p-q; 0)\,dp \Big\}\,dy\end{aligned}$$

where $p \equiv mx/t$, $q \equiv my/t$ and $\alpha \equiv ma/t$. But generally

$$\begin{aligned}\int_a^b f(p-q)\,dp &= \int_{-b}^{-a} f(-p-q)\,dp \\ &= \pm \int_{-b}^{-a} f(p+q)\,dp \quad \text{if } f(\bullet) \text{ is } \begin{cases}\text{even}\\ \text{odd}\end{cases}\end{aligned}$$

so <u>if $P(x,p;0)$ were an even function of momentum</u> then

$$\begin{aligned}\Big\{\text{etc.}\Big\} &= \Big[\cdots + \int_{-3\alpha}^{-2\alpha} + \int_{-2\alpha}^{-\alpha} + \int_{-\alpha}^{0} + \int_0^{\alpha} + \int_{\alpha}^{2\alpha} + \int_{2\alpha}^{3\alpha} \Big] P(y, p+q; 0)\,dp \\ &= \int_{-\infty}^{+\infty} P(y, p+q; 0)\,dp = \int_{-\infty}^{+\infty} P(y, p; 0)\,dp\end{aligned}$$

would give

$$\int_0^a p(x,t)\,dx = \int_0^a \Big\{ \int_{-\infty}^{+\infty} P(y,p;0)\,dp \Big\} dy = 1$$